FROM THE MIDPOINT CONVEXITY TO FULL CONVEXITY

Abstract

We will prove that midpoint convexity implies full convexity under mild conditions.

Notes

First, let's define these terms. We say a function $f : (0, 1) \to \mathbb{R}$ is λ-convex if

$$f(\lambda x + (1 - \lambda)y) \le \lambda f(x) + (1 - \lambda)f(y), \quad \forall x, y \in (0, 1).$$

If f satisfies this condition for $\lambda = 1/2$, it is called midpoint convex. If f is λ-convex for any $\lambda \in (0, 1)$, then it is fully convex.

To proceed, we define the set of dyadic numbers (or dyadic rationals), denoted by $\mathbb{D}$, as the set of rational numbers of the form:

$$\mathbb{D} = \cup_{n=0}^{\infty} \mathbb{D}_n,$$

where

$$\mathbb{D}_n = \left\{ \frac{k}{2^n} \;\middle|\; k = 0, \dots, 2^n \right\}.$$

Proposition 0.1. *If $f : (0, 1) \to \mathbb{R}$ satisfies midpoint convexity and is locally bounded above, then*

(1) f is locally bounded;
(2) f is λ-convex for any dyadic number λ;
(3) f is locally Lipschitz;
(4) f is fully convex.

Proof. (1) **Local boundedness:** Since f is locally upper bounded, it's enough to show that f is locally lower bounded. Fix $x_0 \in (0, 1)$. We set

$$\delta = \min\{x_0, 1 - x_0\}/2.$$

By assumption, there exists $M \in \mathbb{R}$ such that

$$M = \sup_{x \in (x_0 - \delta, x_0 + \delta)} f(x) < +\infty.$$

For any $x \in (x_0 - \delta, x_0 + \delta)$, we set $y = 2x_0 - x$. Note that $y \in (x_0 - \delta, x_0 + \delta)$ as well. By midpoint convexity, we have

$$f(x) \ge 2f(x_0) - f(y) \ge 2f(x_0) - M.$$

This shows that f is locally lower bounded on $(0, 1)$.

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(2) **Dyadic convexity:** Let G be the set of $\lambda \in [0,1]$ such that f is λ-convex. Immediately from the definition, we have $\mathbb{D}_0 = \{0,1\} \subset G$. Assume that $\mathbb{D}_n \subset G$ for some $n \geq 0$. We will show that $\mathbb{D}_{n+1} \subset G$. Indeed, if $\lambda_1, \lambda_2 \in G$, then by midpoint convexity, we have $\frac{\lambda_1+\lambda_2}{2} \in G$ as well. Since any element in $\mathbb{D}_{n+1}$ is the average of two elements in $\mathbb{D}_n$, we conclude that $\mathbb{D}_{n+1} \subset G$. By induction, we have $\mathbb{D}_n \subset G$ for all $n \geq 0$, and hence $\mathbb{D} \subset G$.

(3) **Local Lipschitz continuity:** Fix $x_0 \in (0,1)$. We set

$$\delta = \min\{x_0, 1-x_0\}/4.$$

By local boundedness, there exists $M > 0$ such that

$$|f(x)| \leq M, \quad \forall x \in (x_0 - 2\delta, x_0 + 2\delta).$$

Now, we show that f is Lipschitz on $(x_0 - \delta, x_0 + \delta)$ with a bounded Lipschitz constant. Choose any $x < y \in (x_0 - \delta, x_0 + \delta)$. Let $z_n = \max\{y + \delta_n | \delta_n \in \mathbb{D}_n \cap (0,\delta)\}$. Note that

$$z_n = y + \delta_n \to y + \delta = z < x_0 + 2\delta \text{ as } n \to \infty.$$

By dyadic convexity, we have

$$\frac{f(y) - f(x)}{y - x} \leq \frac{f(z_n) - f(y)}{z_n - y} \leq \frac{1}{\delta_n} \cdot 2M \to \frac{2M}{\delta}, \text{ as } n \to \infty.$$

This shows the upper bound of Lipschitz constant on $(x_0 - \delta, x_0 + \delta)$. Similarly, we can take $w_n = \min\{x - \delta_n | \delta_n \in \mathbb{D}_n \cap (0,\delta)\}$. Then, we have

$$\frac{f(x) - f(y)}{x - y} \geq \frac{f(x) - f(w_n)}{x - w_n} \geq \frac{1}{\delta_n} \cdot (-2M) \to \frac{-2M}{\delta}, \text{ as } n \to \infty.$$

This shows the lower bound of Lipschitz constant on $(x_0 - \delta, x_0 + \delta)$.

(4) **Full convexity:** Full convexity follows from continuity and dyadic convexity.

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